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Discrete monotone method for space-fractional nonlinear reaction–diffusion equations

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Abstract

A discrete monotone iterative method is reported here to solve a space-fractional nonlinear diffusion–reaction equation. More precisely, we propose a Crank–Nicolson discretization of a reaction–diffusion system with fractional spatial derivative of the Riesz type. The finite-difference scheme is based on the use of fractional-order centered differences, and it is solved using a monotone iterative technique. The existence and uniqueness of solutions of the numerical model are analyzed using this approach, along with the technique of upper and lower solutions. This methodology is employed also to prove the main numerical properties of the technique, namely, the consistency, stability, and convergence. As an application, the particular case of the space-fractional Fisher's equation is theoretically analyzed in full detail. In that case, the monotone iterative method guarantees the preservation of the positivity and the boundedness of the numerical approximations. Various numerical examples are provided to illustrate the validity of the numerical approximations. More precisely, we provide an extensive series of comparisons against other numerical methods available in the literature, we show detailed numerical analyses of convergence in time and in space against fractional and integer-order models, and we provide studies on the robustness and the numerical performance of the discrete monotone method.

MSC: Primary 65M06; secondary 35K15; 35K55; 35K57

Keywords: Space-fractional diffusion–reaction equations; Crank–Nicolson finite-difference scheme; Discrete monotone iterative method; Existence and uniqueness of solutions; Numerical efficiency analysis

1 Introduction

Monotone iterative methods have been used in the literature to investigate differential equations (ordinary or partial) from both the analytical and numerical points of view. For example, from the analytical side, such iterative techniques have been applied to investigate the existence and uniqueness of solutions of a wide range of parabolic partial differential equations [1], as well as other analytical features of the solutions. In particular, this approach has been used to establish the existence of positive solutions of quasilinear parabolic systems with Dirichlet boundary conditions [2], to study quasilinear parabolic and elliptic systems with mixed quasimonotone functions [3], to analyze periodic boundary-value problems for differential equations with delay [4], to solve first-order functional-difference equations with nonlinear boundary value conditions [5], to prove the existence and asymptotic behavior of solutions for quasilinear parabolic systems [6], and, recently, to establish the existence, uniqueness, and stability of the solutions of a parabolic model in the formation of porous silicon [7], among other interesting applications.

From the numerical point of view, monotone iterative methods have been also employed to effectively solve systems consisting of differential equations. As for the continuous case, the numerical monotone iterative methods require the knowledge of upper and lower solutions in order to generate two monotone sequences that converge to the solution of the problems under investigation. Numerical techniques of this nature have been employed to solve the multidimensional semiconductor Poisson equation [8], to simulate quantum-corrected energy transport models [9], to study numerically the solutions of parabolic problems with time delays [10], to investigate two-dimensional simulation of submicron MOSFETs [11], to provide numerical analysis of coupled systems of nonlinear parabolic equations [12], and to simulate porous silicon morphologies [13]. In various of these reports and many other articles which employ discrete monotone iterative approaches, this methodology has been used to prove the existence and uniqueness of solutions, as well as to investigate the numerical efficiency of the computational algorithms.

In summary, the monotone method has been extensively employed in the analysis and simulation of nonlinear systems of parabolic differential equations. Moreover, this method has been extended to investigate fractional systems of differential equations. Indeed, in recent years, fractional calculus has found a wide range of applications to viscoelasticity [14], the discretization of nonsingular Mittag-Leffler kernels [15], and fractional operators with exponential kernels and a Lyapunov type inequality [16] among other problems. Furthermore, it has been proved that some families of equations with long-range interactions lead to models governed by fractional differential equations in the continuous limit [17, 18]. In summary, fractional calculus has experienced fast development in the last years, and the development of monotone iterative techniques has seen continuous development within that area of research. However, many interesting problems still remain open to this day. One of them is the development of discrete monotone iterative methods to solve space-fractional diffusion–reaction regimes that generalize the well-known Fisher's equation [19, 20]. This model is the simplest diffusive model with nonlinear reaction, and its many generalizations have been a highly transited avenue of research in mathematics and numerical analysis.

Fractional forms of Fisher's equation have been investigated numerically in various works, considering various generalizations and following different approaches [21]. Indeed, some bounded schemes have been recently proposed to solve multidimensional problems with anomalous diffusion [22], some dynamically consistent methods have been designed also to solve advection–reaction systems with fractional diffusion [23]. On the other hand, there are various extensions of the discrete monotone iterative method to solve fractional differential equations. For example, there are reports on the monotone iterative method for ordinary differential equations involving Riemann–Liouville fractional derivatives [24]. Other articles employ this approach to solve nonlinear fractional q-difference equations with integral boundary conditions [25], others apply it for Riemann–Liouville fractional integro-differential equations with advanced arguments [26], and some others use it to solve Riesz space distributed-order advection–dispersion equations [27]. Various works report on the design of monotone methods to solve fractional diffusion equations with Caputo fractional derivatives in time [28, 29]. However, the Riesz space-fractional scenario has been left without study, perhaps in light of the difficulties arising in such a case.

The novelty of the present work lies in the fact that a monotone iterative method will be proposed for the first time in the literature to solve parabolic partial differential equations with Riesz fractional diffusion. We consider in this case a nonlinear reaction term, so that the mathematical model under investigation is a fractional extension of the well-known Fisher's equation from population dynamics. Suitable initial-boundary conditions will be imposed on a closed and bounded interval of the real numbers. In a first step, we will propose a Crank–Nicolson discretization of the fractional system, and a monotone iterative method will be proposed to solve the discretized model. Existence and uniqueness of solutions will hinge on the fact that the discrete model can be rewritten in vector form, and that the associated coefficient matrix is an M-matrix [30]. Moreover, the consistency, stability and quadratic convergence of the technique will be mathematically proved using the monotone iterative approach, by imposing suitable computational requirements. A particularly meaningful form of the fractional model will be investigated mathematically in deeper detail, and conditions for the existence, uniqueness, and numerical efficiency of the discrete solutions will be derived in that particular case.

This manuscript is organized as follows. In Sect. 2, we introduce the fractional diffusive problem on interest and provide a Crank–Nicolson discretization based on the use of fractional-order centered differences. Here, it is worthwhile to recall that fractional-order centered differences have been used successfully by some of the authors in the discretization of various fractional problems [31–33]. Suitable discrete nomenclature is introduced to that end, including some helpful properties of the fractional centered differences and a convenient vector representation of the numerical model. Section 3 is devoted to present the discrete monotone method for the numerical method of this work. The existence and uniqueness of solutions of the iterative method are thoroughly proved in that stage. The most important numerical properties of the methodology are established in Sect. 4, while Sect. 5 is devoted to theoretically analyze the numerical methodology for the fractional Fisher's equation. Also, we provide thorough numerical comparisons of our methodology against various other approaches proposed in the literature. In particular, we show detailed numerical analyses of convergence in time and in space against fractional and integer-order models, and we provide studies on the robustness and numerical performance of the discrete monotone method. Section 6 is devoted to discuss the findings of this work. Finally, we close this manuscript with a section of concluding remarks.

2 Preliminaries

Throughout this work, we let $L, T \in \mathbb{R}^+$ and $\alpha \in (0,1) \cup (1,2]$. Consider an open spatial domain $\Omega = (0,L) \subseteq \mathbb{R}$, and define the set $\Omega_T = \Omega \times (0,T)$. In this manuscript, we let $v : \overline{\Omega}_T \to \mathbb{R}$ be a sufficiently differentiable function. Define v as zero outside of Ω_T, and we suppose that it satisfies the initial-boundary-value problem

$$\begin{aligned} &\frac{\partial v(x,t)}{\partial t} - K \frac{\partial^\alpha v(x,t)}{\partial |x|^\alpha} = f\big(x,t,v(x,t)\big), \quad \forall (x,t) \in \Omega_T, \\ &\text{such that } \begin{cases} v(x,0) = v_0(x), & \forall x \in \Omega, \\ v(x,t) = 0, & \forall (x,t) \in \partial\Omega \times [0,T]. \end{cases} \end{aligned} \tag{1}$$

The fractional derivative in (1) is understood here in the Riesz sense, that is, we let

$$\frac{\partial^\alpha v(x,t)}{\partial |x|^\alpha} = -\frac{1}{2\cos(\frac{\alpha\pi}{2})\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{v(t,\xi)\,d\xi}{|x-\xi|^{\alpha-1}}, \quad \forall (x,t)\in\Omega_T. \tag{2}$$

Here, Γ is the usual Gamma function and $v_0 : \Omega \to \mathbb{R}$ is sufficiently smooth.

Definition 1 A function $\tilde{v} \in C^{2,1}(\Omega_T)\cap C(\overline{\Omega}_T)$ is called *an upper solution* of (1) if it solves the problem

$$\begin{aligned} &\frac{\partial \tilde{v}(x,t)}{\partial t} - K\frac{\partial^\alpha \tilde{v}(x,t)}{\partial |x|^\alpha} \ge f\bigl(x,t,\tilde{v}(x,t)\bigr), \quad \forall (x,t)\in\Omega_T, \\ &\quad\text{such that } \begin{cases} \tilde{v}(x,0) \ge v_0(x), & \forall x\in\Omega, \\ \tilde{v}(x,t) \ge 0, & \forall (x,t)\in\partial\Omega\times[0,T]. \end{cases} \end{aligned} \tag{3}$$

Similarly, we say that a function $\hat{v} \in C^{2,1}(\Omega_T)\cap C(\overline{\Omega}_T)$ is *a lower solution* of (1) if it satisfies (3) with all the inequalities reversed. If $\tilde{v}$ and $\hat{v}$ are, respectively, upper and lower solutions of (1) then we will assume that they are *ordered*, that is, they satisfy $\hat{v} \le \tilde{v}$. With this nomenclature, we will suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable, and that there are suitable bounded functions $\underline{c} \equiv \underline{c}(x,t)$ and $\overline{c} \equiv \overline{c}(x,t)$ such that

$$-\underline{c}(x,t)(v_1 - v_2) \le f(x,t,v_1) - f(x,t,v_2) \le \overline{c}(x,t)(v_1 - v_2), \tag{4}$$

for all $\hat{v} \le v_2 \le v_1 \le \tilde{v}$ and $(x,t)\in\Omega_T$.

For convenience, we let $I_P = \{1,\ldots,P\}$ and $\overline{I}_P = I_P \cup \{0\}$, for each $P\in\mathbb{N}$. Let $M,N\in\mathbb{Z}$, and consider a grid of Ω_T using uniform spatial and temporal nodes of the forms $x_i = ih$ and $t_k = k\tau$, respectively, for each $i\in\overline{I}_M$ and $k\in\overline{I}_N$. The spatial and temporal partition norms are $h = L/M$ and $\tau = T/N$, respectively, and we use the symbols v_i^k and V_i^k, respectively, to represent the exact and numerical solutions of (1) at (x_i,t_k). In this work, the temporal partial derivative of (1) will be calculated through the forward-difference scheme

$$\frac{\partial v(x_i,t_k)}{\partial t} = \frac{v_i^{k+1} - v_i^k}{\tau} + \mathcal{O}(\tau), \quad \forall (i,k)\in I_{M-1}\times I_{N-1}. \tag{5}$$

Definition 2 (Ortigueira [34]) For any function $\varphi : \mathbb{R}\to\mathbb{R}$, $h>0$ and $\alpha > -1$ we define the *fractional centered difference* of order α of φ at the point x as

$$\Delta_h^\alpha \varphi(x) = \sum_{m=-\infty}^{\infty} g_m^\alpha \varphi(x - mh), \quad \forall x\in\mathbb{R}, \tag{6}$$

whenever the right-hand side of this expression converges. The coefficients $(g_k^\alpha)_{k=-\infty}^{\infty}$ are defined by

$$g_m^\alpha = \frac{(-1)^m \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2} - m + 1)\Gamma(\frac{\alpha}{2} + m + 1)}, \quad \forall m\in\mathbb{N}\cup\{0\}. \tag{7}$$

Lemma 3 (Wang et al. [27]) *If* $\alpha \in (0,1) \cup (1,2]$ *then the coefficients* (7) *satisfy the following properties*:

(i) $g_0^\alpha > 0$.
(ii) $g_m^\alpha = g_{-m}^\alpha < 0$ *for each* $m \in \mathbb{N}$.
(iii) $\sum_{m=-\infty}^{\infty} g_m^\alpha = 0$. *As a consequence*, $g_0^\alpha = -\sum_{m=-\infty, m\neq 0}^{\infty} g_m^\alpha$.

As a consequence of this lemma, the series in the right-hand side of (6) converges absolutely for any bounded function $\varphi \in L_1(\mathbb{R})$. With this notation, it is easy to see that any $\varphi \in \mathcal{C}^5(\mathbb{R})$, for which all of its derivatives up to order five belong to $L_1(\mathbb{R})$, has the property

$$-\frac{1}{h^\alpha}\Delta_h^\alpha \varphi(x) = \frac{\partial^\alpha \varphi(x)}{\partial |x|^\alpha} + \mathcal{O}(h^2), \quad \forall x \in \mathbb{R}, \tag{8}$$

whenever $\alpha \in (0,1) \cup (1,2]$ (see [27]). Under these circumstances, if $1 \leq j \leq M-1$ and $1 \leq n \leq N-1$ then

$$\frac{\partial^\alpha u}{\partial |x|^\alpha}(x_j, t_n) = -\frac{1}{h^\alpha}\sum_{k=-(b-x_j)/h}^{(x_j-a)/h} g_k^{(\alpha)} u(x_j - kh, t_n) + \mathcal{O}(h^2) = \delta_x^{(\alpha)} u_j^n + \mathcal{O}(h^2), \tag{9}$$

where

$$\delta_x^{(\alpha)} u_j^n = -\frac{1}{h^\alpha}\sum_{k=0}^{M} g_{j-k}^{(\alpha)} u_k^n. \tag{10}$$

In this work, we use formulas (5) and (9) in order to propose the following scheme to solve (1):

$$V_i^{k+1} + \frac{\tau K}{2}\delta_x^{(\alpha)} V_i^{k+1} = V_i^k - \frac{\tau K}{2}\delta_x^{(\alpha)} V_i^k + \frac{\tau}{2}\left[f\left(x_i, t_{k+1}, V_i^{k+1}\right) + f\left(x_i, t_k, V_i^k\right)\right]. \tag{11}$$

It is obvious that this numerical model is a Crank–Nicolson-type of scheme. For the sake of convenience, we let $r = \frac{1}{2}\tau h^{-\alpha}$, and define the $(M+1)$-dimensional real vectors

$$\mathcal{V}_k = \left(V_0^k, V_1^k, \ldots, V_M^k\right), \quad \forall k \in \bar{I}_N, \tag{12}$$

$$F(\mathcal{V}_k) = \left(f\left(x_0, t_k, V_0^k\right), f\left(x_1, t_k, V_1^k\right), \ldots, f\left(x_M, t_k, V_M^k\right)\right), \quad \forall k \in \bar{I}_N. \tag{13}$$

Using this nomenclature, system (11) can be readily rewritten in vector form as

$$(I + rKA)\mathcal{V}_{k+1} = (I - rKA)\mathcal{V}_k + rh^\alpha\left[F(\mathcal{V}_{k+1}) + F(\mathcal{V}_k)\right], \tag{14}$$

where I is the identity matrix of size $(M+1) \times (M+1)$, and A is the matrix of the same size of I given by

$$A = \begin{bmatrix} g_0^\alpha & g_{-1}^\alpha & g_{-2}^\alpha & \cdots & g_{-M}^\alpha \\ g_1^\alpha & g_0^\alpha & g_{-1}^\alpha & \cdots & g_{1-M}^\alpha \\ g_2^\alpha & g_1^\alpha & g_0^\alpha & \cdots & g_{2-M}^\alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_M^\alpha & g_{M-1}^\alpha & g_{M-2}^\alpha & \cdots & g_0^\alpha \end{bmatrix}. \tag{15}$$

3 Discrete monotone method

The purpose of the present section is to introduce a discrete monotone method, and use it to establish the existence and uniqueness of the solutions of the discrete system (14). In the following, given any real vectors $\mathcal{V}$ and $\mathcal{W}$ of the same dimensions, we will employ the notation $\mathcal{V} \leq \mathcal{W}$ (alternatively, $\mathcal{W} \geq \mathcal{V}$) meaning that each component of $\mathcal{V}$ is less than or equal to the corresponding component of $\mathcal{W}$.

Let $\underline{c}(x,t)$ and $\overline{c}(x,t)$ be bounded functions satisfying the requirements in Definition 1. Introduce now the real diagonal matrices of sizes $(M+1)\times(M+1)$

$$\underline{C} = \begin{bmatrix} \underline{c}_0^k & 0 & \dots & 0 \\ 0 & \underline{c}_1^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \underline{c}_M^k \end{bmatrix}, \qquad \overline{C} = \begin{bmatrix} \overline{c}_0^k & 0 & \dots & 0 \\ 0 & \overline{c}_1^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \overline{c}_M^k \end{bmatrix}, \tag{16}$$

satisfying $B = I + r(KA + h^\alpha \underline{C})$, $D = I - r(KA + h^\alpha \underline{C})$ and $\mathcal{F}(\mathcal{V}_k) = \underline{C}\mathcal{V}_k + F(\mathcal{V}_k)$. Here, we have omitted the dependence of $\overline{C}$ and $\underline{C}$ on k for the sake of convenience. Note that (4) implies that $\mathcal{F}(\mathcal{V}_k)$ is nondecreasing. Using Picard iterations, we reach the iterative system

$$\begin{cases} B\overline{\mathcal{V}}_{k+1}^{(p)} = D\overline{\mathcal{V}}_k^{(p)} + rh^\alpha[\mathcal{F}(\overline{\mathcal{V}}_{k+1}^{(p-1)}) + \mathcal{F}(\overline{\mathcal{V}}_k^{(p-1)})], & \forall k \in \overline{I}_{N-1}, \\ B\underline{\mathcal{V}}_{k+1}^{(p)} = D\underline{\mathcal{V}}_k^{(p)} + rh^\alpha[\mathcal{F}(\underline{\mathcal{V}}_{k+1}^{(p-1)}) + \mathcal{F}(\underline{\mathcal{V}}_k^{(p-1)})], & \forall k \in \overline{I}_{N-1}, \\ \overline{\mathcal{V}}_0^{(p)} = V_0 = \underline{\mathcal{V}}_0^{(p)}. \end{cases} \tag{17}$$

The initial iteration is given by $(\overline{\mathcal{V}}_k^{(0)}, \underline{\mathcal{V}}_k^{(0)}) = (\tilde{\mathcal{V}}_k, \hat{\mathcal{V}}_k)$, where $\tilde{\mathcal{V}}_k$ and $\hat{\mathcal{V}}_k$ are, respectively, ordered upper and lower solutions of (14), in the sense of Definition 4 below. In the sequel, system (17) will be referred to simply as the *discrete monotone* (DM) method.

Definition 4 The $(M+1)$-dimensional real vector $\tilde{\mathcal{V}}_k$ is an *upper solution* of (14) if it satisfies the inequalities

$$\begin{aligned} &(I + rKA)\tilde{\mathcal{V}}_{k+1} \geq (I - rKA)\tilde{\mathcal{V}}_k + rh^\alpha\left[F(\tilde{\mathcal{V}}_{k+1}) + F(\tilde{\mathcal{V}}_k)\right], \quad \forall k \in \overline{I}_{N-1}, \\ &\tilde{\mathcal{V}}_0 \geq V_0. \end{aligned} \tag{18}$$

Similarly, the real vector $\hat{\mathcal{V}}_k$ is a *lower solution* of (14) if it satisfies all the reversed inequalities in (18). If $\{\tilde{\mathcal{V}}_k\}_{k=0}^N$ and $\{\hat{\mathcal{V}}_k\}_{k=0}^N$ are respectively sets of upper and lower solutions of (14), we will assume tacitly that they are *ordered*, that is, $\tilde{\mathcal{V}}_k \geq \hat{\mathcal{V}}_k$ for all $k \in \overline{I}_{N-1}$.

Definition 5 A square real matrix A is an *M-matrix* if there exist a nonnegative matrix B and a number $\mu \geq \rho(B)$ such that A can be expressed in the form $A = \mu I - B$. Here, $\rho(B)$ represents the spectral radius of B.

It is important to recall that M-matrices are nonsingular, and their inverses are nonnegative matrices.

Theorem 6 (Plemmons [35]) *A square real matrix A is an M-matrix if and only if*

(i) *all its diagonal entries are positive,*

(ii) *all its off-diagonal components are nonpositive, and*
(iii) *there exists a diagonal matrix D with positive diagonal entries such that AD is strictly diagonally dominant.*

Lemma 7 *The matrix $\Xi = KA + h^\alpha \underline{C}$ is an M-matrix if*

$$h^\alpha \max_{0 \le s \le M} \left|\underline{c}_s^k\right| < K \sum_{k=0}^{M} g_k^\alpha. \tag{19}$$

Proof The diagonal components of the matrix Ξ are given by $\xi_{s,s} = K g_0^\alpha + h^\alpha \underline{c}_s^k$, for $s \in \overline{I}_M$. The hypothesis (19) and Lemma 3 guarantee then that (i) of Theorem 6 is satisfied. On the other hand, note that the off-diagonal elements $\xi_{r,s}$ of Ξ are the constants g_k^α, for suitable indexes $k \in \mathbb{Z} \setminus \{0\}$, whence condition (ii) of Theorem 6 holds. Finally, (19) assures that Ξ is already diagonally dominant in view of

$$\sum_{s=0}^{M} \xi_{r,s} = K \sum_{s=0}^{M} g_{r-s} + h^\alpha \underline{c}_s^k, \quad \forall r \in \overline{I}_M. \tag{20}$$

The conclusion of the theorem readily follows now from Theorem 6. □

Theorem 8 (Existence and uniqueness) *Let $\tilde{\mathcal{V}}_k$ and $\hat{\mathcal{V}}_k$ be respectively upper and lower solutions of* (14) *at time t_k, for each $k \in I_N$. Define the constant*

$$c = \max_{1 \le k \le N} \left\{ \left(\max_{0 \le s \le M} \left|\underline{c}_s^k\right| \right) \vee \left(\max_{0 \le s \le M} \left|\overline{c}_s^k\right| \right) \right\}, \tag{21}$$

and, for each $k \in I_N$, let $\{\overline{\mathcal{V}}_k^{(p)}\}_{p=0}^\infty$ and $\{\underline{\mathcal{V}}_k^{(p)}\}_{p=0}^\infty$ be the sequences generated by the DM method (17), *with initial iterations $\overline{\mathcal{V}}_k^{(0)} = \tilde{\mathcal{V}}_k$ and $\underline{\mathcal{V}}_k^{(0)} = \hat{\mathcal{V}}_k$. If the condition*

$$0 < r \left(K \sum_{k=0}^{M} g_k^\alpha - h^\alpha c \right) < 1 \tag{22}$$

is satisfied then these sequences have the property that

$$\hat{\mathcal{V}}_k = \underline{\mathcal{V}}_k^{(0)} \le \underline{\mathcal{V}}_k^{(1)} \le \cdots \le \underline{\mathcal{V}}_k^{(p)} \le \overline{\mathcal{V}}_k^{(p)} \le \cdots \le \overline{\mathcal{V}}_k^{(1)} \le \overline{\mathcal{V}}_k^{(0)} = \tilde{\mathcal{V}}_k, \quad \forall p \in \mathbb{N} \cup \{0\}. \tag{23}$$

Moreover, the sequences converge to the unique solution of (14) *between its upper and lower solutions.*

Proof We will prove the theorem in three steps. Firstly, we will show inductively that (23) is satisfied, for each $k \in I_N$ and $p \in \mathbb{N} \cup \{0\}$. Secondly, we will prove that the sequences converge to the solution of (14) and, finally, we will establish the existence and uniqueness of solutions as a direct consequence of the first and second steps.

1. Note that $\underline{\mathcal{W}}_k^{(1)} = \underline{\mathcal{V}}_k^{(1)} - \underline{\mathcal{V}}_k^{(0)} = \underline{\mathcal{V}}_k^{(1)} - \hat{\mathcal{V}}_k$. Using inequality (18), and adding and subtracting the terms $rh^\alpha \underline{C} \hat{\mathcal{V}}_{k+1}$ and $rh^\alpha \underline{C} \hat{\mathcal{V}}_k$, respectively, it follows that

$$B\hat{\mathcal{V}}_{k+1} \le D\hat{\mathcal{V}}_k + rh^\alpha [\underline{C}\hat{\mathcal{V}}_{k+1} + \underline{C}\hat{\mathcal{V}}_k] + rh^\alpha \left[F(\hat{\mathcal{V}}_{k+1}) + F(\hat{\mathcal{V}}_k)\right]. \tag{24}$$

We subtract then inequality (24) from the following equation that results from (17):

$$B\underline{\mathcal{V}}_{k+1}^{(1)} = D\underline{\mathcal{V}}_k^{(1)} + rh^{\alpha}\big[\mathcal{F}\big(\underline{\mathcal{V}}_{k+1}^{(0)}\big) + \mathcal{F}\big(\underline{\mathcal{V}}_k^{(0)}\big)\big]. \tag{25}$$

As a consequence, we obtain $B\underline{\mathcal{W}}_{k+1}^{(1)} = B\underline{\mathcal{V}}_{k+1}^{(1)} - B\hat{\mathcal{V}}_k \geq D\underline{\mathcal{W}}_k^{(1)}$. But B^{-1} exists and is nonnegative, which implies that $\underline{\mathcal{W}}_{k+1}^{(1)} \geq B^{-1}D\underline{\mathcal{W}}_k^{(1)}$. On the other hand, system (17) implies that $\underline{\mathcal{W}}_1^{(1)} \geq B^{-1}D\underline{\mathcal{W}}_0^{(1)} = 0$. Inductively, if $\underline{\mathcal{W}}_k^{(1)} \geq 0$ for $k \in \overline{I}_{N-1}$, then $\underline{\mathcal{W}}_{k+1}^{(1)} \geq B^{-1}D\underline{\mathcal{W}}_k^{(1)} \geq 0$ and $\underline{\mathcal{V}}_k^{(1)} \geq \underline{\mathcal{V}}_k^{(0)}$ are also satisfied. In summary, we have established so far that $\underline{\mathcal{W}}_k^{(1)} \geq 0$, for each $k \in \overline{I}_N$.

Let $p \in \mathbb{N}$, assume that $\underline{\mathcal{V}}_k^{(p)} \geq \underline{\mathcal{V}}_k^{(p-1)}$ and let $\underline{\mathcal{W}}_k^{(p+1)} = \underline{\mathcal{V}}_k^{(p+1)} - \underline{\mathcal{V}}_k^{(p)}$. From system (17) we obtain

$$B\underline{\mathcal{W}}_k^{(p+1)} = D\underline{\mathcal{W}}_k^{(p+1)} + rh^{\alpha}\big[\mathcal{F}\big(\underline{\mathcal{V}}_{k+1}^{(p)}\big) - \mathcal{F}\big(\underline{\mathcal{V}}_{k+1}^{(p-1)}\big) + \mathcal{F}\big(\underline{\mathcal{V}}_k^{(p)}\big) - \mathcal{F}\big(\underline{\mathcal{V}}_k^{(p-1)}\big)\big]. \tag{26}$$

As a consequence of $\mathcal{F}$ being monotone nondecreasing, $B\underline{\mathcal{W}}_{k+1}^{(p+1)} \geq D\underline{\mathcal{W}}_k^{(p+1)}$ and $\underline{\mathcal{W}}_{k+1}^{(p+1)} \geq B^{-1}D\underline{\mathcal{W}}_k^{(p+1)}$. An inductive argument over k and reuse of the arguments employed above leads to $\underline{\mathcal{W}}_{k+1}^{(p+1)} \geq 0$ and $\underline{\mathcal{V}}_k^{(p+1)} \geq \underline{\mathcal{V}}_k^{(p)}$. The proofs that $\overline{\mathcal{V}}_k^{(p)} \geq \overline{\mathcal{V}}_k^{(p+1)}$ and $\overline{\mathcal{V}}_k^{(p)} \geq \underline{\mathcal{V}}_k^{(p)}$ are obtained in a similar fashion, letting $\overline{\mathcal{W}}_k^{(p+1)} = \overline{\mathcal{V}}_k^{(p)} - \overline{\mathcal{V}}_k^{(p+1)}$ and $\mathcal{W}_k^{(p)} = \overline{\mathcal{V}}_k^{(p)} - \underline{\mathcal{V}}_k^{(p)}$. We have established thus the validity of the chain of inequalities (23).

2. Expression (23) implies that $\{\overline{V}_k^{(p)}\}_{p=0}^{\infty}$ is nonincreasing and bounded from below, while $\{\underline{V}_k^{(p)}\}_{p=0}^{\infty}$ is nondecreasing and bounded from above. This implies that the following limits exist for each $k \in \overline{I}_N$:

$$\lim_{p\to\infty} \overline{V}_k^{(p)} = \overline{\mathcal{V}}_k, \qquad \lim_{p\to\infty} \underline{V}_k^{(p)} = \underline{\mathcal{V}}_k. \tag{27}$$

Obviously, $\overline{\mathcal{V}}_k$ and $\underline{\mathcal{V}}_k$ are solutions of the difference system (14). Now, let $\mathcal{V}_k^*$ be another solution of (14), and define $\mathcal{W}_k = \overline{\mathcal{V}}_k - \mathcal{V}_k^*$. Using (14), then

$$B\mathcal{W}_{k+1} = D\mathcal{W}_k + rh^{\alpha}\big[F(\overline{\mathcal{V}}_{k+1}) - F\big(\mathcal{V}_{k+1}^*\big) + F(\overline{\mathcal{V}}_k) - F\big(\mathcal{V}_k^*\big)\big]. \tag{28}$$

Note that $\mathcal{V}_k^*$ is also a lower solution of (14), so $\mathcal{W}_k \geq 0$. As consequences of inequalities (23), it follows that $F(\overline{\mathcal{V}}_k) - F(\mathcal{V}_k^*) \leq \overline{C}(\overline{\mathcal{V}}_k - \mathcal{V}_k^*))$ and

$$(I + rKA)\mathcal{W}_{k+1} \leq (I - rKA)\mathcal{W}_k + rh^{\alpha}[\overline{C}\mathcal{W}_{k+1} + \overline{C}\mathcal{W}_k], \tag{29}$$

which implies that

$$\big[I + r\big(KA - h^{\alpha}\overline{C}\big)\big]\mathcal{W}_{k+1} \leq \big[I - r\big(KA - h^{\alpha}\overline{C}\big)\big]\mathcal{W}_k. \tag{30}$$

On the other hand, hypothesis (22) implies that $Q = KA - h^{\alpha}\overline{C}$ is an M-matrix, and we know that the matrix $I - rQ$ is positive. Therefore, we reach $\mathcal{W}_{k+1} \leq (I + rQ)^{-1}(I - rQ)\mathcal{W}_k$ in light that each solution of (14) satisfies the initial condition, that is, $\overline{\mathcal{V}}_0 = \mathcal{V}_0^*$. It follows that $W_1 \leq 0$ and, using induction over k, we obtain that $\mathcal{W}_k \leq 0$ and $\overline{\mathcal{V}}_k = \mathcal{V}_k^*$. The proof that $\mathcal{V}_k^* = \underline{\mathcal{V}}_k$ is analogous, letting $\mathcal{Z}_k = \mathcal{V}^* - \underline{\mathcal{V}}_k$.

3. Finally, the construction of the sequence $\{\mathcal{V}_k\}$ can be readily established using induction over k. The approximation $\mathcal{V}_0$ is defined through the initial condition. Supposing that $\mathcal{V}_k$ has been already obtained, $\mathcal{V}_{k+1}$ is reached using the iterative technique and the approximation $\mathcal{V}_k$. □

4 Numerical efficiency

In this stage, we establish the main numerical properties of the DM method, namely, the consistency, stability and convergence. In our proofs, we will require that the conditions of Theorem 8 are satisfied in order to guarantee the existence and uniqueness of the solutions of the DM method.

Definition 9 Let $\{Y_k\}$ and $\{\overline{Y}_k\}$ be sequences of approximations to the solution of the problem $Ay = 0$, obtained by using the scheme $Ay_k = 0$ with the initial conditions Y_0 and $\overline{Y}_0$, respectively. The difference scheme is *stable* if given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|Y_k - \overline{Y}_k\| < \varepsilon$ for all $k \in I_N$, whenever $\|Y_0 - \overline{Y}_0\| \leq \delta$.

We will need the following result to prove the stability of method (14).

Lemma 10 (Flores and Jerez [13]) *Let $\{Y_{k+1}^{(r)}\}$ be a sequence of vectors with $\|Y_0^{(r)}\|, \|Y_k^{(0)}\| \leq \delta$, such that*

$$\left\|Y_{k+1}^{(r)}\right\| \leq a_1\left\|Y_k^{(r)}\right\| + a_2\left\|Y_{k+1}^{(r-1)}\right\| + a_3\left\|Y_k^{(r-1)}\right\|, \tag{31}$$

where a_1, a_2 and a_3 are positive constants with $a_1 < 1$. Then, for all $k \in \overline{I}_R$,

$$\left\|Y_{k+1}^{(r)}\right\| \leq \left[\vartheta^r + (1 + a_3)\left(\vartheta^r + \vartheta^{r-1} + \cdots + \vartheta + 1\right)\right]\delta, \tag{32}$$

where $\vartheta = \frac{a_2 + a_3}{1 - a_1}$.

Theorem 11 *The DM method is unconditionally stable if* (22) *is satisfied.*

Proof Suppose that $\delta > 0$, and let $\mathcal{Z}_k^{(p)}$ be a perturbation of $\overline{\mathcal{V}}_k^{(p)}$ or $\underline{\mathcal{V}}_k^{(p)}$, with the property that $\|\mathcal{Z}_0^{(p)} - \overline{\mathcal{V}}_0^{(p)}\| < \delta$ and $\|\mathcal{Z}_k^{(0)} - \overline{\mathcal{V}}_k^{(0)}\| < \delta$. Suppose that $\mathcal{Z}_k^{(p)}$ satisfies

$$B\mathcal{Z}_{k+1}^{(p)} = D\mathcal{Z}_k^{(p)} + rh^\alpha\left[\mathcal{F}\left(\mathcal{Z}_{k+1}^{(p-1)}\right) + \mathcal{F}\left(\mathcal{Z}_k^{(p-1)}\right)\right]. \tag{33}$$

Define $\mathcal{Y}_k^{(p)} = \mathcal{Z}_k^{(p)} - \overline{\mathcal{V}}_k^{(p)}$, so $\|\mathcal{Y}_0^{(p)}\| < \delta$ and $\|\mathcal{Y}_k^{(0)}\| < \delta$. Subtracting (17) from (33), we obtain

$$\begin{aligned} B\mathcal{Y}_{k+1}^{(p)} = D\mathcal{Y}_k^{(p)} &+ rh^\alpha\left[\mathcal{F}\left(\mathcal{Z}_{k+1}^{(p-1)}\right) - \mathcal{F}\left(\mathcal{V}_{k+1}^{(p-1)}\right)\right] \\ &+ rh^\alpha\left[\mathcal{F}\left(\mathcal{Z}_k^{(p-1)}\right) - \mathcal{F}\left(\mathcal{V}_k^{(p-1)}\right)\right] \end{aligned} \tag{34}$$

and

$$\left\|\mathcal{Y}_{k+1}^{(p)}\right\| \leq a_1\left\|\mathcal{Y}_k^{(p)}\right\| + a_2\left\|\mathcal{Y}_{k+1}^{(p-1)}\right\| + a_3\left\|\mathcal{Y}_k^{(p-1)}\right\|, \tag{35}$$

where $a_1 = \|B^{-1}D\|$, $a_2 = rh^\alpha\|B^{-1}\overline{C}\|$, and $a_3 = rh^\alpha\|B^{-1}\overline{C}\|$. Notice that

$$a_1 = \left\|\left[I + r\left(kA + h^\alpha\underline{C}\right)\right]^{-1}\left[I - r\left(kA + h^\alpha\underline{C}\right)\right]\right\| < 1. \tag{36}$$

Finally, inequality (32) follows by Lemma 10. We conclude that method (17) is unconditionally stable. □

In the previous section, we showed the DM method (17) converges to a unique solution of the discrete system (14). We will prove now that the DM method (17) converges also the solution of the continuous problem (1).

Theorem 12 (Consistency) *If $v(x,t) \in C^{5,2}(\overline{\Omega}_T)$ and f is continuously differentiable then the discrete scheme (14) is consistent with equation (1).*

Proof Let ϕ and ψ be the exact and approximation operators, respectively, corresponding to equation (1). Obviously, these operators satisfy

$$\phi_i^k = \frac{\partial v_i^k}{\partial t} - K\frac{\partial^\alpha v_i^k}{\partial |x|^\alpha} - f\left(x_i, t_k, v_i^k\right), \tag{37}$$

$$\begin{aligned}\psi_i^k = {} & \frac{V_i^{k+1} - V_i^k}{\tau} + \frac{K}{2}\delta_x^{(\alpha)}\left(V_m^{k+1} + V_m^k\right) \\ & - \frac{1}{2}\left[f\left(x_i, t_{k+1}, V_i^{k+1}\right) + f\left(x_i, t_k, V_i^k\right)\right],\end{aligned} \tag{38}$$

for each $(i,k) \in I_{M-1} \times I_{N-1}$. Using the smoothness of the functions v and f, along with the consistency properties of the individual discrete operators and Taylor's theorem, there exist positive constants C_1, C_2, and C_3 such that

$$\left\| \frac{\partial v_i^k}{\partial t} - \frac{V_i^{k+1} - V_i^k}{\tau} \right\| \le C_1\tau, \tag{39}$$

$$\left\| \frac{\partial^\alpha v_i^k}{\partial |x|^\alpha} - \left[-\frac{1}{2}\delta_x^{(\alpha)}\left(V_m^{k+1} + V_m^k\right)\right] \right\| \le C_2\left(\tau + h^2\right), \tag{40}$$

$$\left\| f\left(x_i, t_k, v_i^k\right) - \frac{1}{2}\left[f\left(x_i, t_{k+1}, V_i^{k+1}\right) + f\left(x_i, t_k, V_i^k\right)\right] \right\| \le C_3\tau, \tag{41}$$

for each $(i,k) \in I_{M-1} \times I_{N-1}$. If $C = \max\{C_1, C_2, C_3\}$ then $\|\phi_i^k - \psi_i^k\| \le C(\tau + h^2)$, whence the consistency of (14) readily follows. □

Finally, we prove the convergence of the discrete method.

Theorem 13 (Convergence) *Let $v(x,t) \in C^{5,2}(\overline{\Omega}_T)$, and let f be continuously differentiable. If f satisfies (4) and the hypotheses of Theorem (8) hold then the DM method (17) converges to the unique solution of equation (1).*

Proof Let $\mathcal{V}_k^{(p)}$ be a term of any of the sequences $\{\underline{\mathcal{V}}_k^{(p)}\}$ or $\{\overline{\mathcal{V}}_k^{(p)}\}$ in (17), and let V_k and W_k be the exact solutions of (14) and (1), respectively. We will show firstly that $\|\mathcal{V}_k^{(p)} - W_k\| \to 0$ when $p \to \infty$ and $\tau, h \to 0^+$. Adding and subtracting the term V_k and using the triangle inequality, it follows that $\|\mathcal{V}_k^{(p)} - W_k\| \le \|\mathcal{V}_k^{(p)} - V_k\| + \|V_k - W_k\|$. Theorem 8 yields now that

$$\lim_{p\to\infty} \left\| \mathcal{V}_k^{(p)} - V_k \right\| = 0, \quad \forall k \in \overline{I}_N.$$

It only remains to show that the solutions of the discrete system (14) converge to those of the continuous, that is,

$$\lim_{\tau,h\to 0^+} \|V_k - W_k\| = 0, \quad \forall k \in \overline{I}_N. \tag{42}$$

To that end, let $V_k = (v_0^k, v_1^k, \ldots, v_M^k)$ and $W_k = (w_0^k, w_1^k, \ldots, w_M^k)$. Using Lemma 12, the discrete system (14), hypotheses of this theorem, and matrix $\overline{C}$, we obtain that

$$\begin{aligned}
&(I + rKA)(V_{k+1} - W_{k+1}) + \mathcal{O}(\tau + h^2) \\
&\quad = (I - rKA)(V_k - W_k) + rh^\alpha\big[F(V_{k+1}) - F(W_{k+1}) + F(V_k) - F(W_k)\big] \\
&\quad \le (I - rKA)(V_k - W_k) + rh^\alpha\overline{C}[V_{k+1} - W_{k+1}] + rh^\alpha\overline{C}[V_k - W_k],
\end{aligned} \tag{43}$$

for each $k \in \overline{I}_{N-1}$. Then $[I + rQ](V_{k+1} - W_{k+1}) \le [I - rQ](V_k - W_k) + \mathcal{O}(\tau + h^2)$, where $Q = KA - h^\alpha\overline{C}$. Now, inequality (22) guarantees that the $I + rQ$ is positive. This implies that

$$\|V_{k+1} - W_{k+1}\| \le \|\Lambda\|\|V_k - W_k\| + \|\Theta\|\mathcal{O}(\tau + h^2), \quad \forall k \in \overline{I}_{N-1}, \tag{44}$$

where $\Lambda = [I + rQ]^{-1}[I - rQ]$ and $\Theta = [I + rQ]^{-1}$. Using induction, we obtain

$$\|V_k - W_k\| \le \|\Lambda\|^k\|V_0 - W_0\| + \|\Theta\|\|\Lambda\|^{k-1}\mathcal{O}(\tau + h^2), \quad \forall k \in \overline{I}_{N-1}. \tag{45}$$

The limit (42) is readily established now from the facts that $\|\Lambda\| < 1$ and $\|\Theta\| < 1$. This completes the proof. □

5 Numerical examples

In this section, we describe the computational implementation of the DM method to solve Fisher's equation with fractional diffusion. More concretely, we will consider the problem

$$\begin{aligned}
&\frac{\partial v(x,t)}{\partial t} - K\frac{\partial^\alpha v(x,t)}{\partial|x|^\alpha} = v(x,t)\big[1 - v(x,t)\big], \quad \forall (x,t) \in \Omega_T, \\
&\quad \text{such that } \begin{cases} v(x,0) = v_0(x), & \forall x \in \Omega, \\ v(x,t) = 0, & \forall (x,t) \in \partial\Omega \times [0,T]. \end{cases}
\end{aligned} \tag{46}$$

To that end, note that condition (4) requires for the functions $\underline{c}(x,t)$ and $\overline{c}(x,t)$ to be calculated as

$$\underline{c}(x,t) = \sup\left\{-\frac{\partial f(v)}{\partial v} : \hat{v} \le v \le \tilde{v}\right\}, \tag{47}$$

$$\overline{c}(x,t) = \sup\left\{\frac{\partial f(v)}{\partial v} : \hat{v} \le v \le \tilde{v}\right\}. \tag{48}$$

From equation (46), it follows that $\underline{c}(x,t) = \overline{c}(x,t) = \eta = \max\{2\gamma - 1, 1\}$, where $\gamma = \max\{1, \max_\Omega v_0(x)\}$. Using the uniform grid introduced in Sect. 2 along with the matrix system (17), it follows that

$$B\mathcal{V}_{k+1}^{(p)} = D\mathcal{V}_k^{(p)} + rh^\alpha\big[\mathcal{F}\big(\mathcal{V}_{k+1}^{(p-1)}\big) + \mathcal{F}\big(\mathcal{V}_k^{(p-1)}\big)\big], \tag{49}$$

with

$$B = \begin{bmatrix} 1 + r(Kg_0^\alpha + h^\alpha \eta) & g_{-1}^\alpha & \dots & g_{-M}^\alpha \\ g_1^\alpha & 1 + r(Kg_0^\alpha + h^\alpha \eta) & \dots & g_{1-M}^\alpha \\ g_2^\alpha & g_1^\alpha & \dots & g_{2-M}^\alpha \\ \vdots & \vdots & \ddots & \vdots \\ g_M^\alpha & g_{M-1}^\alpha & \dots & 1 + r(Kg_0^\alpha + h^\alpha \eta) \end{bmatrix}, \tag{50}$$

$$D = \begin{bmatrix} 1 - r(Kg_0^\alpha + h^\alpha \eta) & g_{-1}^\alpha & \dots & g_{-M}^\alpha \\ g_1^\alpha & 1 - r(Kg_0^\alpha + h^\alpha \eta) & \dots & g_{1-M}^\alpha \\ g_2^\alpha & g_1^\alpha & \dots & g_{2-M}^\alpha \\ \vdots & \vdots & \ddots & \vdots \\ g_M^\alpha & g_{M-1}^\alpha & \dots & 1 - r(Kg_0^\alpha + h^\alpha \eta) \end{bmatrix}. \tag{51}$$

On the other hand, the vectors $\mathcal{V}_k$ and $\mathcal{F}(\mathcal{V}_k)$ are given by

$$\mathcal{V}_k = \left(V_0^k, V_1^k, \dots, V_M^k\right)^\top, \tag{52}$$

$$\mathcal{F}(\mathcal{V}_k) = \left(\eta V_0^k + V_0^k\left(1 - V_0^k\right), \eta V_1^k + V_1^k\left(1 - V_1^k\right), \dots, \eta V_M^k + V_0^k\left(1 - V_M^k\right)\right)^\top. \tag{53}$$

Lemma 14 *Let* $\gamma = \max\{1, \max_\Omega v_0(x)\}$. *Then the vectors* $\tilde{\mathcal{V}} = \gamma(1, \dots, 1)^\top$ *and* $\hat{\mathcal{V}} = (0, \dots, 0)^\top$ *of* $\mathbb{R}^{M+1}$ *are respectively upper and lower solutions of the discretization of* (46).

Proof From Lemma 3, it follows that $\sum_{m=0}^M g_{i-m}^\alpha > 0$, for each $i \in \bar{I}_M$. Using system (14), it is clear that the $\tilde{\mathcal{V}}$ satisfies the inequality

$$1 + \gamma r K_1 \sum_{m=0}^M g_{i-m}^\alpha > 1 - \gamma r K_1 \sum_{m=0}^M g_{i-m}^\alpha. \tag{54}$$

On the other hand, for $\hat{\mathcal{V}}$ the equality is satisfied since $F(\hat{\mathcal{V}}) = (0, \dots, 0)^\top$. □

Using the upper and lower solutions provided in Lemma 14, our implementation of the DM method will make use of the iterative system

$$\begin{cases} B\overline{\mathcal{V}}_{k+1}^{(p)} = D\overline{\mathcal{V}}_k^{(p)} + rh^\alpha[\mathcal{F}(\overline{\mathcal{V}}_{k+1}^{(p-1)}) + \mathcal{F}(\overline{\mathcal{V}}_k^{(p-1)})], \\ B\underline{\mathcal{V}}_{k+1}^{(p)} = D\underline{\mathcal{V}}_k^{(p)} + rh^\alpha[\mathcal{F}(\underline{\mathcal{V}}_{k+1}^{(p-1)}) + \mathcal{F}(\underline{\mathcal{V}}_k^{(p-1)})], \\ \overline{\mathcal{V}}_0^{(p)} = (V_0^0, V_1^0, \dots, V_M^0) = \underline{\mathcal{V}}_0^{(p)}, \end{cases} \tag{55}$$

where $(\overline{\mathcal{V}}_k^{(0)}, \underline{\mathcal{V}}_k^{(0)}) = (\tilde{\mathcal{V}}_k, \hat{\mathcal{V}}_k)$. Moreover, the next lemma establishes a condition to ensure the converge.

Lemma 15 *Suppose that the following condition is satisfied*:

$$rK \sum_{k=0}^M g_k^\alpha - 1 < \frac{\eta}{2}\tau < rK \sum_{k=0}^M g_k^\alpha. \tag{56}$$

Then the sequences $\{\overline{\mathcal{V}}_k^{(p)}\}$ *and* $\{\underline{\mathcal{V}}_k^{(p)}\}$ *obtained by* (55) *converge to the unique solution of* (46) *between* $\hat{\mathcal{V}}_k$ *and* $\tilde{\mathcal{V}}_k$.

Proof The proof follows from Theorem 8. □

The following is a trivial consequence of the previous lemma.

Theorem 16 (Positivity and boundedness) *Suppose that condition* (56) *is satisfied. Then the discrete model* (46) *is capable of preserving the positivity and the boundedness from above by* 1 *of the numerical approximations.*

Next, we provide some computational simulations to confirm the validity of the approximations obtained through the DM method. In view of the lack of known exact solutions for the fully fractional model considered in this work, we compare our results with those of other techniques available in the literature. Beforehand, we must mention that our computational implementation of the DM monotone method will hinge on the use of lower and upper solutions for the problem under investigation. They will be used as starting approximations at each iteration in order to generate the sequences (23). As stopping criterion, we will set a maximum difference in the infinity norm equal to 1×10^{-10} or a maximum number of iterations equal to 20. It is important to point out that this maximum number of iterations was never reached in our simulations. In fact, the maximum error was obtained usually in 8 iterations of the DM method.

Example 17 Let $\Omega = (0, \pi)$ and $T = 1$, and we will consider the problem

$$\begin{aligned} &\frac{\partial v(x,t)}{\partial t} - K\frac{\partial^\alpha v(x,t)}{\partial |x|^\alpha} = \sin\bigl(v(x,t)\bigr), \quad \forall (x,t) \in \Omega_T, \\ &\text{such that } \begin{cases} v(x,0) = \sin x, & \forall x \in \Omega, \\ v(x,t) = 0, & \forall (x,t) \in \partial\Omega \times [0,1]. \end{cases} \end{aligned} \tag{57}$$

For convenience, we let $K = 0.1$. Computationally, we fix the parameters $h = \pi/200$ and $\tau = 0.016$. Figure 1 shows the approximate solution of problem (57) at the times $t = 0.3$ (solid), 0.5 (dashed), 1 (dashed–dotted) and 5 (dotted), using $\alpha = 1.2, 1.4, 1.6$, and 1.8. The behavior of the solutions is in qualitative agreement with those results obtained in [36]. Moreover, our simulations show that the solutions tend to a stationary solution as t increases, and that this solution is approximately reached at time $t = 5$. In order to validate the code of our DM method, we will set various combinations of the values of the parameters h and τ, and let $\alpha = 1.8$. For comparisons, we will use the explicit finite-difference approximation (EFDA), the implicit finite-difference approximation (IFDA) method and the fractional method of lines (FMoL) reported in [36], as well as the DM method of this work. The results are provided in Tables 1–3, for the times $T = 0.3, 1$, and 3, respectively. The results indicate that the DM methods yields approximations which are in qualitative agreement with those obtained through the EFDA, IFDA, and FMoL. It is worth mentioning that we have conducted more experiments considering different nodes of Ω and different approximation times. The results obtained using the EFDA, IFDA, FMoL, and DM method (not shown here in order to avoid redundancy) are in qualitative agreement.

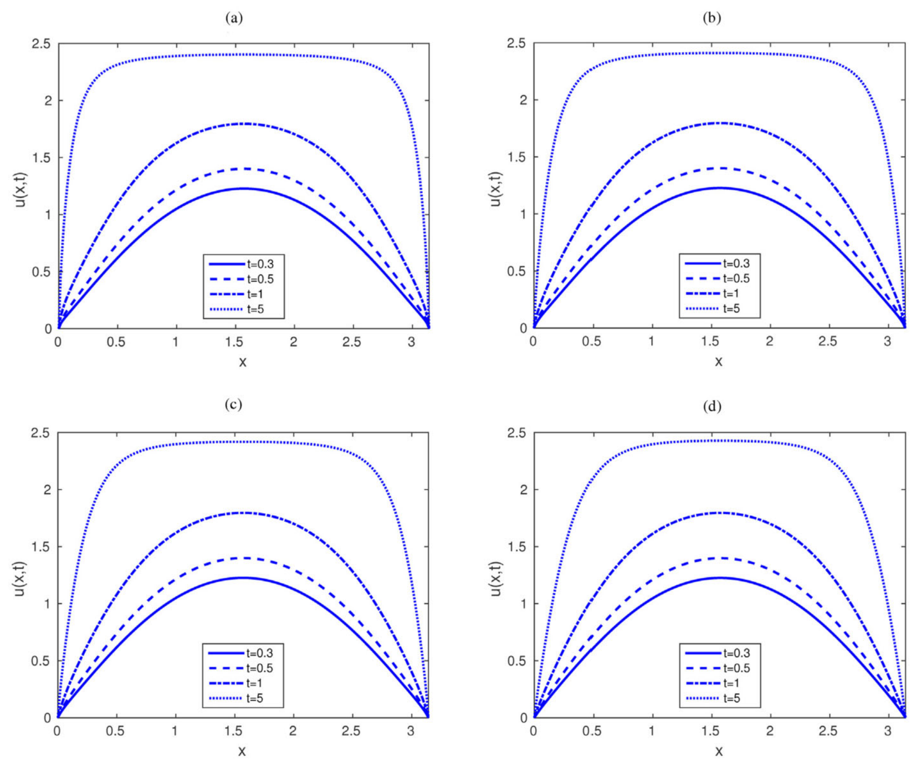

Figure 1 Approximate solution of the problem (57) using the DM method, at the times $t = 0.3$ (solid), 0.5 (dashed), 1 (dashed–dotted) and 5 (dotted). Various values of α were used, namely, (**a**) $\alpha = 1.2$, (**b**) $\alpha = 1.4$, (**c**) $\alpha = 1.6$, and (**d**) $\alpha = 1.8$. The solutions were calculated over $\Omega = (0, \pi)$, and using $K = 0.1$. Computationally, we let $h = \pi/200$ and $\tau = 0.016$

Next, we will perform an analysis of convergence of the DM method, considering normal diffusion and long times. To that end, we will consider problem (46) with $K = 1$, and employ the exact traveling-wave solution

$$v(x,t) = \left[1 + C\exp\left(\frac{1}{\sqrt{6}}x - \frac{5}{6}t\right)\right]^{-2}, \quad \forall (x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}. \tag{58}$$

In our simulations, we will modify the methodology proposed in this work to account for exact Dirichlet boundary conditions, prescribing them through (58). Moreover, we will employ the maximum-norm error between the exact solution of (46) at the time T, and the corresponding numerical solution calculated using the DM method, namely,

$$\epsilon_{\tau,h} = \max\left\{\left|v(x_i, t_K) - V_i^K\right| : i \in \overline{I}_M\right\}. \tag{59}$$

We also define the following standard rates:

$$\rho_\tau = \log_2\left(\frac{\epsilon_{2\tau,h}}{\epsilon_{\tau,h}}\right), \qquad \rho_h = \log_2\left(\frac{\epsilon_{\tau,2h}}{\epsilon_{\tau,h}}\right). \tag{60}$$

Example 18 Consider problem (46) with normal diffusion and $K = 1$. Fix $\Omega = (-200, 200)$, and use function (58) to prescribe exactly the initial and boundary data on Ω. Under these

Table 1 Values of the approximate solution of problem (57) at different values of x and $T = 0.3$, using $\Omega = (0, \pi)$, $\alpha = 1.8$, and $K = 0.1$. Computationally, we used different combinations of values of h and τ. The results were obtained using the EFDA, IFDA, and FMoL reported in [36], as well as the DM method introduced in the present manuscript

	Method			
x	EFDA	IFDA	FMoL	DM method
$h = \pi/200$, $\tau = 0.016$				
0.3142	0.40643185	0.40591753	0.40722026	0.40523810
0.6283	0.75093552	0.75000260	0.75219110	0.74870365
0.9425	1.01391315	1.01236366	1.01519975	1.01087353
1.2566	1.17543235	1.17356309	1.17739215	1.17196504
1.5708	1.22928866	1.22798315	1.23181079	1.22597857
1.8850	1.17482175	1.17399495	1.17730057	1.17196504
2.1991	1.01388718	1.01274958	1.01539033	1.01087353
2.5133	0.75121711	0.75012166	0.75238197	0.74870365
2.8274	0.40650703	0.40608269	0.40746642	0.40523810
$h = \pi/400$, $\tau = 0.008$				
0.3142	0.40798803	0.40800055	0.40803064	0.40790925
0.6283	0.75305289	0.75306981	0.75312693	0.75290589
0.9425	1.01613053	1.01613171	1.01623145	1.01593914
1.2566	1.17754898	1.17754418	1.17768450	1.17733677
1.5708	1.23163392	1.23163641	1.23173842	1.23140839
1.8850	1.17754907	1.17755395	1.17763565	1.17733677
2.1991	1.01614381	1.01611409	1.01622990	1.01593914
2.5133	0.75305404	0.75304748	0.75313106	0.75290589
2.8274	0.40798587	0.40798503	0.40803474	0.40790925
$h = \pi/800$, $\tau = 0.004$				
0.3142	0.40923740	0.40924090	0.40925669	0.40924443
0.6283	0.75497993	0.75499925	0.75502314	0.75500177
0.9425	1.01843426	1.01845552	1.01849042	1.01846144
1.2566	1.17998806	1.18000297	1.18004151	1.18000806
1.5708	1.23410056	1.23411599	1.23414084	1.23410729
1.8850	1.18000780	1.18001024	1.18003866	1.18000806
2.1991	1.01846042	1.01846591	1.01848874	1.01846144
2.5133	0.75499623	0.75500746	0.75502307	0.75500177
2.8274	0.40924316	0.4092445549	0.40925683	0.40924443

circumstances, Table 4 provides a temporal numerical convergence analysis of the DM method at various values of T. The results confirm that the method possesses linear order of temporal convergence, in agreement with Theorem 13. In turn, Table 5 shows the spatial convergence analysis of the DM method. Again, the results confirm the conclusion of Theorem 13.

Finally, we compare the performance and the robustness of the methods used in Example 17 using an exact solution for a fractional problem. To that end, we will follow closely the approach of [37] to prove the robustness of our the DM method. As in that work, we will consider the problem (1) defined over $\Omega_T = (0,1) \times (0,1)$, and fix the reaction function as

$$\begin{aligned} f(x,t,v) = {} & \frac{3K}{4}\left[1 + (2\pi)^\alpha\right]\sin(2\pi x) - \frac{K}{4}\left[1 + (6\pi)^\alpha\right]\sin(6\pi x) \\ & + \alpha t^{\alpha-1}\sin^3(2\pi x) - Kv, \quad \forall (x,t,v) \in \Omega_T \times \mathbb{R}. \end{aligned} \tag{61}$$

In this case, the exact solution of the problem for $\alpha \in (1,2]$ is given by

$$v(x,t) = t^\alpha \sin^3(2\pi x), \quad \forall (x,t) \in (0,1) \times (0,1]. \tag{62}$$

Table 2 Values of the approximate solution of problem (57) at different values of x and $T = 1$, using $\Omega = (0,\pi)$, $\alpha = 1.8$, and $K = 0.1$. Computationally, we used different combinations of values of h and τ. The results were obtained using the EFDA, IFDA, and FMoL reported in [36], as well as the DM method introduced in the present manuscript

	Method			
x	EFDA	IFDA	FMoL	DM method
$h = \pi/200$, $\tau = 0.016$				
0.3142	0.71389626	0.72323636	0.71312860	0.71947583
0.6283	1.24037640	1.24612454	1.24780762	1.24680145
0.9425	1.59030472	1.58819777	1.59121579	1.58946310
1.2566	1.78102704	1.77545588	1.77773300	1.77701456
1.5708	1.83207836	1.83457607	1.83648052	1.83573913
1.8850	1.77206738	1.77552773	1.77783878	1.77713605
2.1991	1.58670064	1.58831821	1.59176807	1.59042740
2.5133	1.24770350	1.24628205	1.25101528	1.24589045
2.8274	0.73840275	0.72338189	0.72768746	0.72902617
$h = \pi/400$, $\tau = 0.008$				
0.3142	0.71745620	0.71869375	0.71739547	0.71825673
0.6283	1.24936024	1.24673018	1.24695142	1.24680021
0.9425	1.58920413	1.58824503	1.59047160	1.58896095
1.2566	1.77024701	1.77730806	1.77781163	1.77740204
1.5708	1.83278407	1.83479204	1.83502709	1.83480527
1.8850	1.77106482	1.77732094	1.77759317	1.77762806
2.1991	1.59178365	1.59003758	1.58984672	1.58824720
2.5133	1.24907602	1.24506232	1.24630048	1.24686015
2.8274	0.71260986	0.71746539	0.71745204	0.71730264
$h = \pi/800$, $\tau = 0.004$				
0.3142	0.71796578	0.71802674	0.71796470	0.71807628
0.6283	1.24570889	1.24680373	1.24670367	1.24678390
0.9425	1.58941153	1.58926493	1.58922784	1.58920036
1.2566	1.77697364	1.77729566	1.77740225	1.77726490
1.5708	1.83511650	1.83472546	1.83480026	1.83478660
1.8850	1.77683902	1.77746225	1.77748929	1.77744028
2.1991	1.58896130	1.58906777	1.58887466	1.58902267
2.5133	1.24620755	1.24649110	1.24657834	1.24679221
2.8274	0.71796274	0.71889936	0.71803795	0.71814852

Example 19 Let $\Omega = (0,1)$, $K = 0.1$ and $T = 1$, and consider problem (1) with reaction function given by (61). For illustration purposes, Fig. 2 shows the approximation to the solution $v(x,t)$ of this problem as a function of x and t. We used the DM method to produce the approximations, fixing $h = \tau = 0.01$. Various values of α were employed, namely, (a) $\alpha = 1.01$, (b) $\alpha = 1.2$, (c) $\alpha = 1.4$, (d) $\alpha = 1.6$, (e) $\alpha = 1.8$, and (f) $\alpha = 2$. Figure 3 provides graphical summaries of the temporal convergence and efficiency analyses of the methods used in Example 19. In these analyses, we employed the exact solution (62) of problem (1) with reaction function (61). The results show that the DM method is a first-order convergent technique in time, which yields smaller errors for fixed values of τ. Moreover, the DM method is a more efficient technique according to out results. These results show that the DM method is a more efficient and robust technique than the EFDA, IFDA, and FMoL. It is worth pointing out that we also carried out analysis of spatial performance and robustness. The results (not shown here in view of the redundancy) yield the same conclusions on the DM method.

Before closing this section, we must declare that the simulations were carried out using an implementation of our method in ©Matlab 8.5.0.197613 (R2015a) on a ©Sony Vaio PCG-5L1P laptop computer with Kubuntu 16.10 as operating system. In terms of compu-

Table 3 Values of the approximate solution of problem (57) at different values of x and $T = 3$, using $\Omega = (0, \pi)$, $\alpha = 1.8$, and $K = 0.1$. Computationally, we used different combinations of values of h and τ. The results were obtained using the EFDA, IFDA, and FMoL reported in [36], as well as the DM method introduced in the present manuscript

	Method			
x	EFDA	IFDA	FMoL	DM method
$h = \pi/200$, $\tau = 0.016$				
0.3142	1.57368789	1.57372511	1.57392168	1.57372919
0.6283	2.16380277	2.16386523	2.16399150	2.16385886
0.9425	2.33655001	2.33663342	2.33667019	2.33661347
1.2566	2.38754579	2.38764530	2.38762011	2.38761337
1.5708	2.39918437	2.39929592	2.39922930	2.39925392
1.8850	2.38753659	2.38766383	2.38756809	2.38761337
2.1991	2.33650970	2.33667395	2.33654916	2.33661347
2.5133	2.16369084	2.16393561	2.16376674	2.16385886
2.8274	1.57350732	1.57381573	1.57361994	1.57372919
$h = \pi/400$, $\tau = 0.008$				
0.3142	1.57438633	1.57437820	1.57435879	1.57435676
0.6283	2.16341461	2.16340869	2.16339364	2.16339412
0.9425	2.33597962	2.33597699	2.33597013	2.33597013
1.2566	2.38702266	2.38702177	2.38701950	2.38701864
1.5708	2.39869212	2.39869098	2.39869138	2.39868980
1.8850	2.38702107	2.38701906	2.38702156	2.38701864
2.1991	2.33597346	2.33597038	2.33597368	2.33597013
2.5133	2.16339880	2.16339599	2.16339738	2.16339412
2.8274	1.57436187	1.57436096	1.57436143	1.57435676
$h = \pi/800$, $\tau = 0.004$				
0.3142	1.57400608	1.57400734	1.57400844	1.57400710
0.6283	2.16262828	2.16262984	2.16263132	2.16263025
0.9425	2.33531356	2.33531372	2.33531405	2.33531388
1.2566	2.38648630	2.38648729	2.38648785	2.38648734
1.5708	2.39820239	2.39820243	2.39820257	2.39820244
1.8850	2.38648651	2.38648731	2.38648785	2.38648734
2.1991	2.33531397	2.33531392	2.33531389	2.33531387
2.5133	2.16263007	2.16263026	2.16263073	2.16263025
2.8274	1.57400875	1.57400804	1.57400742	1.57400710

Table 4 Table of absolute errors in the maximum norm and temporal rates of convergence for various values of the parameters τ and h. We used $f(u) = u(1 - u)$, and the exact solution (58) of model (46). We employed also $\Omega = (-200, 200)$ and various values of T

	$h = 1$		$h = 0.5$		$h = 0.25$	
τ	$\epsilon_{\tau,h}$	ρ_τ	$\epsilon_{\tau,h}$	ρ_τ	$\epsilon_{\tau,h}$	ρ_τ
$T = 1$						
$0.2/2^0$	$2.67829504 \times 10^{-2}$	–	$1.02746832 \times 10^{-2}$	–	$6.86107201 \times 10^{-3}$	–
$0.2/2^1$	$1.39736652 \times 10^{-2}$	0.93860442	$5.19210974 \times 10^{-3}$	0.98470113	$3.27302120 \times 10^{-3}$	1.06781106
$0.2/2^2$	$6.96832376 \times 10^{-3}$	1.00382692	$2.42698608 \times 10^{-3}$	1.09715504	$1.33351771 \times 10^{-3}$	1.29538596
$0.2/2^3$	$2.87272809 \times 10^{-3}$	1.27839021	$1.00672149 \times 10^{-3}$	1.26950121	$5.21640040 \times 10^{-4}$	1.35411047
$0.2/2^4$	$1.27134635 \times 10^{-3}$	1.17606433	$4.28389417 \times 10^{-4}$	1.23266988	$2.19716822 \times 10^{-4}$	1.24740928
$T = 10$						
$0.2/2^0$	$3.61963008 \times 10^{-2}$	–	$1.88368494 \times 10^{-2}$	–	$8.75622547 \times 10^{-3}$	–
$0.2/2^1$	$1.94607648 \times 10^{-2}$	0.89527385	$9.56993064 \times 10^{-3}$	0.97697731	$4.18864496 \times 10^{-3}$	1.06382550
$0.2/2^2$	$8.08362730 \times 10^{-3}$	1.26749370	$4.53126541 \times 10^{-3}$	1.07859447	$1.88939130 \times 10^{-3}$	1.14856208
$0.2/2^3$	$3.64953553 \times 10^{-3}$	1.14728994	$2.03288428 \times 10^{-3}$	1.15638590	$8.88224791 \times 10^{-4}$	1.08892478
$0.2/2^4$	$1.74324598 \times 10^{-3}$	1.06593671	$9.01840414 \times 10^{-4}$	1.17258403	$4.32223831 \times 10^{-4}$	1.03914622
$T = 50$						
$0.2/2^0$	$3.96849776 \times 10^{-2}$	–	$1.67291610 \times 10^{-2}$	–	$9.28669337 \times 10^{-3}$	–
$0.2/2^1$	$2.02824578 \times 10^{-2}$	0.96836050	$8.20627788 \times 10^{-3}$	1.02756518	$4.77745217 \times 10^{-3}$	0.95892357
$0.2/2^2$	$9.28799326 \times 10^{-3}$	1.12679366	$3.90749860 \times 10^{-3}$	1.07048265	$2.17094911 \times 10^{-3}$	1.13791552
$0.2/2^3$	$4.43068565 \times 10^{-3}$	1.06783695	$1.77641225 \times 10^{-3}$	1.13727893	$1.02809201 \times 10^{-3}$	1.07835652
$0.2/2^4$	$2.08820809 \times 10^{-3}$	1.08526449	$7.96579516 \times 10^{-4}$	1.15707614	$5.00554256 \times 10^{-4}$	1.03837103

Table 5 Table of absolute errors in the maximum norm and spatial rates of convergence for various values of the parameters τ and h. We used $f(u) = u(1-u)$, and the exact solution (58) of model (46). We employed also $\Omega = (-200, 200)$ and various values of T

	$\tau = 5 \times 10^{-4}$		$\tau = 2.5 \times 10^{-4}$		$\tau = 1.25 \times 10^{-4}$	
h	$\epsilon_{t,h}$	ρ_h	$\epsilon_{t,h}$	ρ_h	$\epsilon_{t,h}$	ρ_h
$T = 1$						
$2/2^0$	$2.72891640 \times 10^{-3}$	–	$7.38903899 \times 10^{-4}$	–	$1.92801667 \times 10^{-4}$	–
$2/2^1$	$6.08617515 \times 10^{-4}$	2.16472044	$1.53496126 \times 10^{-4}$	2.26718449	$4.23532511 \times 10^{-5}$	2.18657290
$2/2^2$	$1.23264181 \times 10^{-4}$	2.30378222	$3.31398131 \times 10^{-5}$	2.21156488	$9.55526103 \times 10^{-6}$	2.14810553
$2/2^3$	$2.40756903 \times 10^{-5}$	2.35610456	$6.97944970 \times 10^{-6}$	2.24738027	$2.24943767 \times 10^{-6}$	2.08673089
$2/2^4$	$5.35953443 \times 10^{-6}$	2.16739758	$1.57552357 \times 10^{-6}$	2.14728195	$5.00208668 \times 10^{-7}$	2.16896243
$T = 10$						
$2/2^0$	$2.80627843 \times 10^{-3}$	–	$7.48551627 \times 10^{-4}$	–	$2.18637328 \times 10^{-4}$	–
$2/2^1$	$5.82837716 \times 10^{-4}$	2.26749201	$1.66642570 \times 10^{-4}$	2.16734482	$5.11309655 \times 10^{-5}$	2.09627056
$2/2^2$	$1.29375821 \times 10^{-4}$	2.17152622	$3.94539680 \times 10^{-5}$	2.07851469	$1.13824890 \times 10^{-5}$	2.16738120
$2/2^3$	$2.71945685 \times 10^{-5}$	2.25017758	$9.26898876 \times 10^{-6}$	2.08968655	$2.60012534 \times 10^{-6}$	2.13016299
$2/2^4$	$6.35975778 \times 10^{-6}$	2.09627481	$2.13604753 \times 10^{-6}$	2.11746820	$6.21903759 \times 10^{-7}$	2.06381793
$T = 50$						
$2/2^0$	$2.87820224 \times 10^{-3}$	–	$7.53867720 \times 10^{-4}$	–	$2.24422947 \times 10^{-4}$	–
$2/2^1$	$6.50328970 \times 10^{-4}$	2.14592637	$1.66761373 \times 10^{-4}$	2.17652624	$5.03484551 \times 10^{-5}$	2.15620078
$2/2^2$	$1.53265284 \times 10^{-4}$	2.08513874	$3.84354956 \times 10^{-5}$	2.11727398	$1.18548423 \times 10^{-5}$	2.08647103
$2/2^3$	$3.65707445 \times 10^{-5}$	2.06726905	$9.18431827 \times 10^{-6}$	2.06519473	$2.82647735 \times 10^{-6}$	2.06839944
$2/2^4$	$8.84445865 \times 10^{-6}$	2.04784425	$2.22227666 \times 10^{-6}$	2.04713420	$6.81731460 \times 10^{-7}$	2.05172967

tational times, we are aware that better results may be obtained with more modern equipment, more modest Linux/Unix distributions and lower-level programming languages.

6 Discussion

Historically, the DM method has been employed successfully to solve numerically and analytically various types of ordinary and partial differential equations [1–3, 12]. As the theory of fractional calculus developed throughout the years, the need to employ reliable techniques to guarantee the existence and uniqueness of relevant solutions of fractional systems directed the attention of researchers to the methods available for integer-order models. In that way, the DM method found applications in the investigation of differential equations of fractional order in time. As we noted, in the literature there are many reports on the adaptation of the DM method to both ordinary and partial differential equations with temporal derivatives of fractional order [24, 38, 39]. In those models, the fractional temporal derivatives are usually understood in the sense of Caputo or Riemann–Liouville. However, the use of the DM method for the case of Riesz fractional derivatives in space is an open problem which merits attention. In that sense, the present manuscript is one of the first reports in which this problem is tackled satisfactorily.

On the other hand, it is important to point out that the main contributions of this work do not report on the design of novel numerical schemes. Indeed, finite differences are employed to obtain a discrete form of the equation under investigation. Here, it is worth recalling that there are various computational approaches to implement finite-difference schemes, depending on the nature of the numerical model. For example, for nonlinear systems, the Newton of quasi-Newton methods (like the well-known Broyden technique) are some standard approaches to that end [40, 41]. In the case of Newton's method, the Jacobian matrix of the reaction functions and its corresponding inverse need to be calculated at each iteration at a high computational cost. On the other hand, Broyden's method requires the calculation of the approximate inverse of the Jacobian matrix. Moreover, this

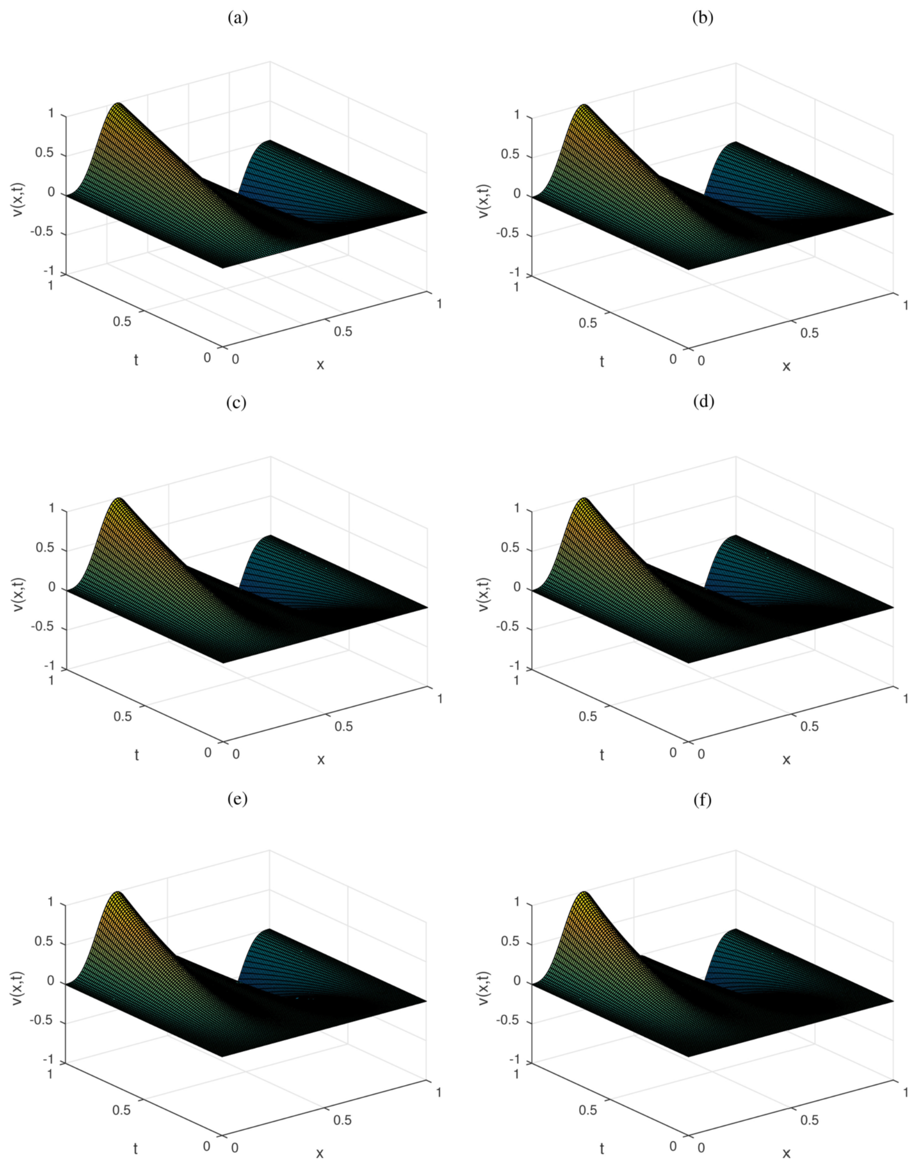

Figure 2 Approximate solution of the problem (1) with reaction function (61) using the DM method, with differentiation order (**a**) $\alpha = 1.01$, (**b**) $\alpha = 1.2$, (**c**) $\alpha = 1.4$, (**d**) $\alpha = 1.6$, (**e**) $\alpha = 1.8$, and (**f**) $\alpha = 2$. The solutions were calculated over $\Omega = (0, 1)$, and using $K = 0.1$ and $T = 1$. Computationally, we let $h = \tau = 0.01$

method is not self-correcting. Other techniques to solve finite-difference schemes are the methods of Gauss–Seidel, Jacobi, and successive over-relaxation, though they are used to solve linear systems of algebraic equations.

In light of these comments, the discrete monotone iterative method is an approach that has been used to solve many finite-difference schemes for both ordinary and partial differential equations of integer order. However, the conditions under which the discrete monotone iterative method guarantees the existence and uniqueness of solutions, stability, and convergence of the computational approach may differ from the respective conditions for the respective finite-difference scheme. This is perhaps one of the reasons why

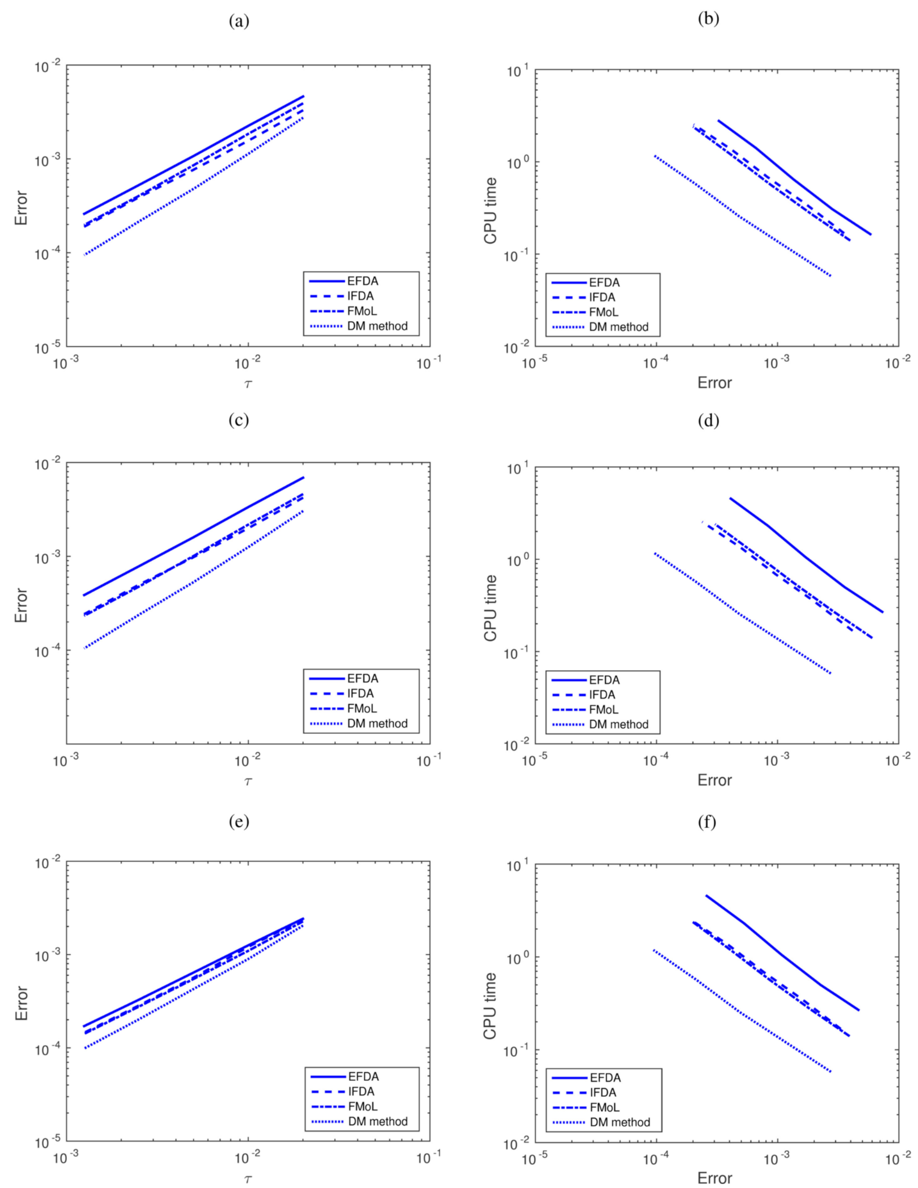

Figure 3 Log–log plots of convergence (left column) and numerical efficiency (right column) of the EFDA (solid), IFDA (dashed), FMoL (dashed–dotted) and DM method (dotted), using various values of α, namely, $\alpha = 1.2$ (top row), $\alpha = 1.4$ (middle row), and $\alpha = 1.8$ (bottom row). The analysis considers problem (1) with reaction function (61), for which the exact solution is given by (62) with $K = 0.1$. For the simulations, we used $\Omega = 1$ and $T = 1$

the literature lacks reports on the discrete monotone iterative method for parabolic partial differential equations with fractional derivatives in space.

Like those computational methods mentioned in the previous paragraphs to solve systems finite-difference equations, the DM method is a computational technique to solve finite-difference schemes. This approach is based on the use of a Picard-like iterative linear system at each iteration. Under suitable conditions on the reaction function and model parameters, the iterative system may generate monotone sequences which converge to the solution of the (nonlinear or linear) problem. To that end, the use of ordered upper and lower solutions of the continuous model is required. Moreover, the implementation of the DM method possess the following advantages:

- The linear character of the DM method can be computationally implemented using iterative techniques for the solution of linear algebraic systems.
- The iterations are monotone sequences, which implies that the error is reduced at each new iteration. Moreover, a suitable criterion of convergence can be readily proposed in terms of the upper and lower solutions at each iteration. In that sense, the method is self-improving.
- Monotonicity of the sequences allows establishing the existence and uniqueness of solutions. This is a clear advantage with respect to arbitrary nonlinear computational methods.
- The solution is bounded between the upper and the lower solutions. This is an important property of the DM method for problems where the positivity and the boundedness are important features of the solutions.
- The theorem on convergence states that the convergence rate of the discrete monotone method is of order $\mathcal{O}(\tau + h^2)$, as expected from the finite-difference discretization. It is important to remember that, in general, the monotone method does not accelerate the convergence rate. The advantage of this iterative method lies in that fewer iterations are required to achieve a certain error level. This feature of our technique was obviously established by our simulations.

7 Conclusions

In this work, for the first time in the literature, the discrete monotone method is developed for reaction–diffusion partial differential equations with fractional diffusion of the Riesz type. The system under investigation considers homogeneous Dirichlet boundary conditions, and is discretized using a Crank–Nicolson technique. The discrete monotone method is used then. We establish that the technique has a unique solution. Moreover, the consistency, the stability and the convergence of the method are rigorously established. The implementation for the case of the space-fractional Fisher's equation is analyzed in detail. We provided an extensive series of comparisons against other numerical methods available in the literature. Moreover, we showed detailed numerical analyses of convergence in time and in space against fractional and integer-order models, and we provided studies on the robustness and the numerical performance of the discrete monotone method.

Before closing this work, it is important to point out that still many avenues of research remain open after the completion of this article. For example, the investigation of more complicated parabolic systems with fractional diffusion in space is still an open problem in investigation. Indeed, there exist many generalizations of the classical Fisher's equation which consider the presence of advection/convection terms, like the Burgers–Fisher and the Burgers–Huxley equations [42]. In that sense, this manuscript could be a motivation to propose and analyze monotone iterative techniques to solve more general fractional parabolic system. On the other hand, it is worthwhile to mention that recent papers have focused on some meaningful applications of hyperbolic fractional systems to the investigation of systems of long-range interactions [43, 44] and fractional [45–48] and even in nonfractional wave equations [49–51]. In that sense, the use of monotone iterative techniques may find interesting applications to the investigation of meaningful physical phenomena.

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Availability of data and materials
The data will not be available online. However, the information will be available to the interested parties upon request.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The research problem was proposed by SF and JEMD. The theoretical analysis was performed by SF and JEMD. The simulations were produced by SF and JEMD. The manuscript was prepared by SF and JEMD, and was later revised and corrected by SF and JEMD. The corrections of the revision were proposed by SF and JEMD. The final revised paper was proposed by SF and JEMD. All authors read and approved the final manuscript.

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